Multi-objective fuzzy optimization for portfolio selection: an embedding theorem approach

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Abstract

The optimal solution of the crisp optimization problem deduced from the fuzzy optimization problem by using embedding theorem is an optimal solution of the original fuzzy optimization problem under the set of core values of fuzzy numbers. In portfolio selection problem, the expected return, risk, liquidity cannot be predicted accurately. So the investor generally makes his portfolio choice according to his knowledge and his economic wisdom. Thus, deterministic portfolio selection is not a sensible option for the investor. Because of the existence of several non-stochastic factors in stock market, fuzzy portfolio selection models have been proposed by many authors. This paper extends the classical Mean-Variance (MV) portfolio selection model into Mean-Variance-Skewness (MVS) model in fuzzy environment under the criteria on short and long term returns, liquidity and dividends. The embedding theorem is used to convert the fuzzy MVS model into crisp multi-objective constrained optimization problem. Finally, optimum solutions of the corresponding crisp multi-objective constrained optimization problem are derived by using global criteria method. This solution is also an optimal solution of the original fuzzy portfolio selection problem. Lastly, our approaches are tested on a set of stock data from Bombay Stock Exchange (BSE), India.

Keywords: Fuzzy portfolio selection, Mean-variance-skewness model, Fuzzy embedding theorem, Trapezoidal fuzzy numbers, Global criteria method

1. Introduction

The embedding problem in topology concerns when a space $X$ can be embedded into another space $Y$, i.e., when there exists an embedding from $X$ into $Y$. Theorems asserting the embedding of a space into some other space which is more manageable than the
original space are known as embedding theorems. On the other hand, a theorem which asserts that a certain space cannot be embedded into some other space is known as non-embedding theorems. Non-embedding theorems are often quite deep and require methods well beyond the general topology. For example it is by no means trivial to prove that the 2-sphere \( S^2 \) cannot be embedded into the Euclidean space. The fuzzy embedding theorem shows that each fuzzy number can be identified isometrically and isomorphically with an element in \( C[0, 1] \times C[0, 1] \) where \( C[0, 1] \) is the set of all real valued bounded functions on \( [0, 1] \). The theorem can be used for multi-objective decision making and has successfully been applied to optimization problems. Some research works on embedding theorem have already been done. Puri and Ralescu (1983) and Kaleva (1990) have proved that the set of all fuzzy numbers can be embedded into a Banach space isometrically and isomorphically. Wu and Ma (1991) provide a specific Banach space, which shows that the set of all fuzzy numbers can be embedded into the Banach space \( C[0, 1] \times C[0, 1] \). Inspired by this specific Banach space, a fuzzy optimization problem can be transformed into a multi-objective optimization problem which can be solved by using interactive fuzzy decision making procedure. Wu (2004) proposes an \((\alpha, \beta)\)-optimal solution concept of fuzzy optimization problem based on the possibility and necessity measures. To do so, the fuzzy optimization problem is transformed into a bi-objective programming problem by applying the embedding theorem. Wu (2004) uses the notions of possibility and necessity to introduce the ranking method on the set of fuzzy numbers. Under this setting, \( \alpha \)-optimal solution concept of fuzzy optimization problem is provided by applying the theorem. Wu (2004) shows that the optimal solution of the crisp optimization problem obtained from the fuzzy optimization problem by using embedding theorem is also an optimal solution of the original fuzzy optimization problem under the set of core values of fuzzy numbers.

While the basis of contemporary mathematical models in finances can be traced back to Bachelier’s (1900) thesis on the theory of speculation, with no dithering, the work of Markowitz (1952) in portfolio selection has been the most impact-making progress in mathematical investment management. In view of the fact that returns are vague in nature, the allotment of capital in different risky assets to minimize the risk and to make the most of the return is the main concern of portfolio selection. Most of the reasonable works on portfolio selection have been done based on merely the first two moments of return distribution. The first order moment about the origin, i.e., the mean, quantifies the return and the second order moment about the mean, i.e., the variance, quantifies the risk. A natural extension of the MV model is to add the skewness as a factor for concern in portfolio management. The third order moment about the mean of a return distribution i.e., skewness measures the asymmetry of the distribution. One interested in considering skewness prefers a portfolio with a higher probability of large payoffs when mean and variance remain the same. The significance of upper order moments in portfolio selection was recommended by Samuelson (1958). But considerations of skewness in portfolio selection problem was started by 1990 and Lai (1991), Konno and Shirakawa (1993), Konno and Suzuki (1995), Inuiguchi and Ramik (2000), Chunhachinda et al. (1997), Liu et al. (2003), Prakash et al. (2003), Giove et al. (2006), Briec et al. (2007), Yu et al. (2008) and others contributed significantly in this context. All the above said literatures assume that the security returns are random variables. But there are many non-stochastic factors that affect stock markets and they are improper to deal with probability approaches. By incurring fuzzy approaches quantitative analysis,
qualitative analysis, experts’ knowledge and investors’ subjective opinions can be better integrated into a portfolio selection model. Ramaswamy (1998), Tanaka and Guo (1999), Inuiguchi and Ramik (2000), Parra et al. (2001), Ida (2004), Zhang and Nie (2004), Terol et al. (2006), Abiyev and Menekay (2007), Vercher et al. (2007), Gupta et al. (2008), Huang (2006, 2008, 2008), Lin and Liu (2008), Li et al. (2010), Bhattacharyya et al. (2009, 2010, 2011) and others study fuzzy portfolio selection. All the relevant information for an investment decision cannot be confined in terms of explicit return, asymmetry and risk. By capturing additional and alternative decision criteria, a portfolio that is dominated with respect to expected return and risk may frame for the shortfall in these two important factors by a very good act on one or several other criteria. As a result, portfolio selection models that consider more criteria than the standard expected return and variance objectives of the Markowitz model (1952) have become well-liked. Arenas et al. (2001) anticipates the model that consists of three criteria, return, risk and liquidity. Ehrgott et al. (2004) propose a model having five criteria, viz., short and long term returns, dividend, ranking and risk and use multi-criteria decision making approach to solve the portfolio selection problem. Fang et al. (2006) propose a portfolio rebalancing model with transaction costs based on fuzzy decision theory considering three criteria, return, risk and liquidity. In this work, the fuzzy MVS portfolio selection model is consiere with constraints on short and long term returns, liquidity and dividends.

The rest of the paper is organized as follows. A discussion on preliminary and backgrounds is given in section 2. In section 3, the optimization model of fuzzy MVS portfolio selection problem with constraints on short and long term returns, liquidity and dividends is constructed by applying fuzzy embedding theorem. In section 4, global criteria optimization technique for solution of the portfolio selection is briefly discussed. In section 5 share price data from Bombay stock market (BSE index), India is used to illustrate the effectiveness of the algorithm and finally section 6 draws some general conclusions.

2. Preliminary and backgrounds

In this section, we present some important definitions, propositions and theorems that will be used in the following sections.

**Definition 2.1:** [Joshi (2004), pp. 121] Let \((X, \tau), (Y, \mathcal{U})\) be two topological spaces. An embedding (or imbedding) theorem of \(X\) into \(Y\) is a function \(e: X \rightarrow Y\) which is a homeomorphism when considered as a function from \((X, \tau)\) onto \((e(X), \mathcal{U}/ e(X))\).

**Proposition 2.2:** [Joshi (2004), pp. 344] A function \(e: X \rightarrow Y\) is an embedding function if and only if it is continuous and one-one and for every open set \(V\) in \(X\) there exists an open subset \(W\) of \(Y\) such that \(e(V) = W \cap Y\).

**Definition 2.3:** [Lee (2005), pp. 129- 137] to qualify as a fuzzy number, a fuzzy set \(\tilde{A}\) on \(\mathbb{R}\) must possess at least the following three properties:

1. \(\tilde{A}\) must be a normal fuzzy set,
b) \( A_\alpha \) must be a closed interval for every \( \alpha \in (0, 1] \),

c) the support of \( \tilde{A} \) must be bounded.

Since \( \alpha \)-cuts of any fuzzy number are required to be closed intervals for all \( \alpha \in (0, 1] \), every fuzzy number is a convex fuzzy set. The inverse is not necessarily true.

\( \tilde{A} \) is a fuzzy number if and only if there exists closed interval \([a, b] \neq \emptyset \) such that

\[
\mu_\alpha(x) = \begin{cases} 
    l(x) & \text{for } x \in (-\infty, a) \\
    1 & \text{for } x \in [a, b] \\
    r(x) & \text{for } x \in (b, \infty)
\end{cases}
\]

where \([a, b]\) is the core of \( \tilde{A} \), \( l \) is a function from \((-\infty, a)\) to \([0, 1]\), that is monotonic increasing, continuous from the right, and is such that \( l(x) = 0 \) for \( x \in (-\infty, a - w_1) \); \( r \) is a function from \((b, \infty)\) to \([0, 1]\), that is monotonic decreasing, continuous from the left, and such that \( r(x) = 0 \) for \( x \in (b + w_2, \infty) \).

Basically this form allows us to define fuzzy numbers in a piecewise manner. We call this fuzzy number of the type LR and denote it by the notation \( \tilde{A} = (a, b, w_1, w_2)_L^R \).

A fuzzy number \( \tilde{A} \) is called positive, symbolized by \( \text{sgn}(\tilde{A}) = +1 \) if

\( \text{Supp}(\tilde{A}) \subseteq ]0, \infty[ \).

A fuzzy number \( \tilde{A} \) is called negative, symbolized by \( \text{sgn}(\tilde{A}) = -1 \) if

\( \text{Supp}(\tilde{A}) \subseteq ]-\infty, 0[ \).

A fuzzy number \( \tilde{A} \) is a fuzzy zero, symbolized by \( \text{sgn}(\tilde{A}) = 0 \) if

\( 0 \in \text{Supp}(\tilde{A}) \).

Let \( \tilde{A} = (a, b, c, d) \) be a trapezoidal fuzzy number with membership function \( \mu_\tilde{A}(x) \) defined by

\[
\mu_\tilde{A}(x) = \begin{cases} 
    \frac{x - a}{b - a} & x \in [a, b] \\
    1 & x \in [b, c] \\
    \frac{d - x}{d - c} & x \in [c, d] \\
    0 & \text{otherwise}
\end{cases}
\]

Its \( \alpha \)-level sets are \( \{\tilde{A}\}_\alpha = [a(\alpha), d(\alpha)] = [a + (b-a)\alpha, d - (d-c)\alpha] \).

Note that if \( b = c \), then \( \mu_\tilde{A}(x) \) is membership function of triangular fuzzy number \( \tilde{A} = (a, b, d) \).

Now by condition b of Definition 2.3, we can write \( \tilde{A}_\alpha = [\tilde{A}_\alpha^L, \tilde{A}_\alpha^U] \).

We use the notation \( \tilde{I}(m) \) to represent the crisp number with value \( m \). Clearly,

\[
[\tilde{I}(m)]_\alpha^L = [\tilde{I}(m)]_\alpha^U = m \forall \alpha \in [0,1].
\]
Definition 2.4: [Puri and Ralesku (1983)] Let $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^n$. The Hausdorff metric is then defined by
\[
d_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \right\}.
\]
The metric $d_H$ on $\mathbb{F}(\mathbb{R})$ is now defined as
\[
d_H(A, B) = \max \left\{ \sup_{0 \leq \alpha \leq 1} d_H(\bar{A}_\alpha, \bar{B}_\alpha), \sup_{0 \leq \alpha \leq 1} d_H(\bar{A}_\alpha, \bar{B}_\alpha) \right\}.
\]
For $\bar{A}, \bar{B}$ in $\mathbb{F}(\mathbb{R})$, we have
\[
d_H(\bar{A}_\alpha, \bar{B}_\alpha) = \max \left\{ \bar{A}_\alpha - \bar{B}_\alpha, \bar{A}_\alpha - \bar{B}_\alpha \right\}.
\]

Definition 2.5: The space $C[0, 1]$ is the set of all real valued bounded functions $f$ on $[0, 1]$, such that $f$ is left-continuous for any $t \in (0, 1]$ and right-continuous at $0$, and $f$ has a right limit for any $t \in [0, 1)$. $C[0, 1]$ is a Banach space with respect to the norm $\|f\| = \sup_{t \in [0, 1]} |f(t)|$.

Definition 2.6: [Wu and Ma (1991)] Let $\bar{A}$ be a fuzzy number in $\mathbb{F}(\mathbb{R})$ and we write $A^L(\alpha) = \bar{A}_\alpha^L$ and $A^U(\alpha) = \bar{A}_\alpha^U$ as the functions of $\alpha \in [0, 1]$. We define the embedding function $\pi: \mathbb{F}(\mathbb{R}) \to C[0, 1] \times C[0, 1]$ by $\pi(\bar{A}) = (A^L(\alpha), A^U(\alpha))$ and state and prove the following theorem:

Theorem 2.7 (Embedding Theorem): Let the function $\pi$ be the embedding function defined by Definition 2.6. Then the following properties hold good.

i) $\pi$ is injective.

ii) $\pi((I(s) \times \bar{A}) + (I(t) \times \bar{B})) = s\pi(\bar{A}) + t\pi(\bar{B}) \forall \bar{A}, \bar{B} \in \mathbb{F}(\mathbb{R}), s \geq 0, t \geq 0$.

iii) $d_F(\bar{A}, \bar{B}) = \|\pi(\bar{A}) - \pi(\bar{B})\|$. That is, $\mathbb{F}(\mathbb{R})$ can be embedded into $C[0, 1] \times C[0, 1]$ isometrically and isomorphically.

Proof:

(i) Let, if possible, $\bar{A}, \bar{B}$ be two distinct fuzzy numbers such that $\pi(\bar{A}) = \pi(\bar{B})$. Then $A^L(\alpha) = B^L(\alpha)$ and $A^U(\alpha) = B^U(\alpha)$, i.e., $(A^L(\alpha), A^U(\alpha)) = (B^L(\alpha), B^U(\alpha))$. Since the two real open intervals are equal, the corresponding boundary points must be the same. Then $A^L(\alpha) = B^L(\alpha)$ and $A^U(\alpha) = B^U(\alpha)$ which contradicts the fact that $\bar{A} \neq \bar{B}$. Hence our assumption is wrong and consequently the mapping $\pi$ is injective.

(ii) We have,
\[
\pi((I(s) \times \bar{A}) + (I(t) \times \bar{B})) = ((I(s) \times \bar{A}) + (I(t) \times \bar{B})) L \alpha, ((I(s) \times \bar{A}) + (I(t) \times \bar{B})) U \alpha \\
= (\{I(s) \times \bar{A}\} L \alpha + \{I(t) \times \bar{B}\} L \alpha, \{I(s) \times \bar{A}\} U \alpha + \{I(t) \times \bar{B}\} U \alpha) \\
= (s\bar{A}_\alpha^L + t\bar{B}_\alpha^L, s\bar{A}_\alpha^U + t\bar{B}_\alpha^U) = s\pi(\bar{A}) + t\pi(\bar{B}).
\]

(iii) We have,
\[ d_\mathcal{E}(\tilde{A}, \tilde{B}) \]
\[ = \sup_{0 \leq \alpha \leq 1} d_H(\tilde{A}_\alpha, \tilde{B}_\alpha) \]
\[ = \sup_{0 \leq \alpha \leq 1} \{ \max \{ |\tilde{A}_\alpha^L - \tilde{B}_\alpha^L|, |\tilde{A}_\alpha^U - \tilde{B}_\alpha^U| \} \} \]
\[ = \max \{ \sup_{0 \leq \alpha \leq 1} |\tilde{A}_\alpha^L - \tilde{B}_\alpha^L|, \sup_{0 \leq \alpha \leq 1} |\tilde{A}_\alpha^U - \tilde{B}_\alpha^U| \} \]
\[ = \|\pi(\tilde{A}) - \pi(\tilde{B})\|_\infty. \]

**Note:** [Kaleva (1990)] The theorem shows that each element in \( \mathbb{F}(\Re) \) can be identified with an element in \( C[0,1] \times C[0,1] \). More specifically, each element \( \tilde{A} \) in \( \mathbb{F}(\Re) \) can be identified with an element \((A^L(\alpha), A^U(\alpha))\) in \( C[0,1] \times C[0,1] \), where \( A^L(\alpha) = \tilde{A}_\alpha^L \), \( A^U(\alpha) = \tilde{A}_\alpha^U \), and this identification is isometric and isomorphic.

**Definition 2.8:** [Wu (2004)] Let \( \tilde{A}, \tilde{B} \in \mathbb{F}(\Re) \). We write \( \tilde{B} \succeq \tilde{A} \) if and only if \( \tilde{B}_\alpha^L \geq \tilde{A}_\alpha^L \) and \( \tilde{B}_\alpha^U \geq \tilde{A}_\alpha^U \) for all \( \alpha \in (0,1) \). Then "\( \succeq \)" is a partial ordering on \( \mathbb{F}(\Re) \). We also write \( \tilde{A} \preceq \tilde{B} \) if and only if \( \tilde{B} \succeq \tilde{A} \). On the other hand, we write that \( \tilde{A} \prec \tilde{B} \) if and only if
\[
\begin{cases} 
\tilde{A}_\alpha^L < \tilde{B}_\alpha^L & \forall \alpha, \\
\tilde{A}_\alpha^U \leq \tilde{B}_\alpha^U & \forall \alpha,
\end{cases}
\]
or
\[
\begin{cases} 
\tilde{A}_\alpha^L \leq \tilde{B}_\alpha^L & \forall \alpha, \\
\tilde{A}_\alpha^U < \tilde{B}_\alpha^U & \forall \alpha.
\end{cases}
\]

We also write \( \tilde{A} \succ \tilde{B} \) if and only if \( \tilde{B} \prec \tilde{A} \).

Let \( f_1, f_2, g_1, g_2 \) be real valued functions defined on the same vector space \( V \). We say that for any fixed \( x_0 \in V \), \( (f_1, g_1) \leq_{x_0} (f_2, g_2) \) if and only if
\[
(f_1(x_0) \leq f_2(x_0), g_1(x_0) \leq g_2(x_0)).
\]

We say that \( (f_1, g_1) <_{x_0} (f_2, g_2) \) if and only if
\[
\begin{cases} 
f_1(x_0) < f_2(x_0), \\
g_1(x_0) \leq g_2(x_0)
\end{cases} \quad \text{or} \quad \begin{cases} 
f_1(x_0) \leq f_2(x_0), \\
g_1(x_0) < g_2(x_0)
\end{cases}
\]

**Proposition 2.9:** [Wu (2004)] Let \( \tilde{A}, \tilde{B} \in \mathbb{F}(\Re) \) and \( \pi \) be the embedding function defined in Definition 2.6. Then \( \tilde{A} \preceq_{\alpha} \tilde{B} \) if and only if \( \pi(\tilde{A}) \preceq_{\alpha} \pi(\tilde{B}) \) for \( \alpha \geq \frac{1}{2} \). We also have \( \tilde{A} \prec_{\alpha} \tilde{B} \) if and only if \( \pi(\tilde{A}) \prec_{\alpha} \pi(\tilde{B}) \) for \( \alpha \geq \frac{1}{2} \).

Let \( \tilde{f} \) be a function defined by \( \tilde{f}: V \to \mathbb{F}(\Re) \), where \( V \) is a real vector space. Then \( \tilde{f} \) is called a fuzzy valued function defined on the real vector space.

Now we consider the following fuzzy constrained multi-objective optimization problem:
where $\tilde{f}_i$ are fuzzy valued functions ($i = 1, 2, 3, ..., n$) and $g_j$ ($j = 1, 2, ..., m$) are real valued functions defined on the same vector space $V$ and $X$ is any subspace of the real vector space $V$.

We say that $x^*$ is an optimal solution of problem (2.1) if there exists no $x \neq x^*$ such that $\tilde{f}_i(x) < (\tilde{f}_i(x^*))$.

Let $\pi$ be the embedding function defined in Definition 2.6. Then we consider the following multi-objective optimization problem by applying the embedding function $\pi$ to problem (2.1).

$$\begin{align*}
\min \{ \pi(\tilde{f}_1(x), \tilde{f}_2(x), ..., \tilde{f}_n(x)) \} \\
= \min \{ \pi(\tilde{f}_1(x)), \pi(\tilde{f}_2(x)), ..., \pi(\tilde{f}_n(x)) \} \\
= \min \{ [(\tilde{f}_1(x))^L,\alpha], [(\tilde{f}_2(x))^L,\alpha], [(\tilde{f}_3(x))^H,\alpha], ..., [(\tilde{f}_n(x))^L,\alpha] \} \\
= \min \{ f_1^L(x,\alpha), f_1^U(x,\alpha), f_2^L(x,\alpha), f_2^U(x,\alpha), ..., f_n^L(x,\alpha), f_n^U(x,\alpha) \} \\
\text{subject to } g_j(x) \leq 0, j = 1,2,\ldots,m \\
x \in X, 0 \leq \alpha \leq 1
\end{align*}$$

We say that $x^*$ is an optimal solution of problem (2.2), if there exists no $x \neq x^*$ such that $\pi(\tilde{f}_i(x)) < \pi([\tilde{f}_i(x^*)]), i.e., (f_1^L(x,\alpha), f_1^U(x,\alpha)) < (f_1^L(x^*,\alpha), f_1^U(x^*,\alpha))$.

**Theorem 2.10:** [Wu (2004)] If $(x^*, \alpha^*)$ is a Pareto optimal solution of (2.1) for some $\alpha^* \in [0,1]$, then $x^*$ is an optimal solution of the fuzzy multi-objective optimization problem (2.2).

**Note:** To solve fuzzy multi-objective problem by using embedding theorem we first have to transform the fuzzy multi-objective optimization problem into the crisp multi-objective optimization problem (2.2). The Pareto optimal solution of this problem is the optimal solution of the original fuzzy multi-objective problem.

**Definition 2.11:** [Lai and Hwang (1994), pp. 28] The positive – ideal solution (PIS) of a multi-objective decision making problem is one that optimizes each objective functions simultaneously.

For example, let $f_a^* = \max_{x \in X} f_a(x), a \in A$ (A is an index set) for maximization objectives

and $f_b^* = \min_{x \in X} f_b(x), b \in B$ (B is an index set) for minimization objective functions. PIS then can be defined as $f^* = \{f_1^*, f_2^*, ..., f_n^* \}$ where $A \cup B = \{1,2,\ldots,n \}$. 

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**Definition 2.12:** [Lai and Hwang (1994), pp. 29] The negative – ideal solution (NIS) of a multi-objective decision making problem is one that yields feasible but the worst solution of each objective functions simultaneously.

For example, let \( f_a^- = \min_{x \in X} f_a(x), a \in A \) (A is an index set) for maximization objectives and \( f_b^- = \max_{x \in X} f_b(x), b \in B \) (B is an index set) for minimization objective functions. PIS then can be defined as \( f^- = \{ f_1^-, f_2^-, \ldots, f_n^- \} \) where \( A \cup B = \{1, 2, \ldots, n\} \).

### 3. The portfolio selection model

In this section we will first describe the assumptions and notations used in the construction of the paper. Then the objective function of the models will be constructed in the next subsection. In the third subsection we will discuss the constraints used in our portfolio selection model. The fourth subsection will include three different mathematical models for different situations.

#### 3.1 Assumptions and notations

Let us consider a financial market with \( n \) risky assets offering random rates of return. An investor allocates his wealth among the risky assets.

For \( i^{th} \) risky asset \((i = 1, 2, \ldots, n)\), let us use the following notations:
- \( x_i = \) the proportion invested;
- \( r_i = \) the random rate of return;
- \( R_i = E(r_i), \) the expected rate of return;
- \( \sigma_{ij} = \text{cov}(r_i, r_j), \) the covariance between \( r_i \) and \( r_j, j = 1, 2, 3, \ldots, n \) and \( (\sigma_{ij})_{n \times n} \) which is semi-positive definite;
- \( d_i = \) the annual dividend;
- \( R_i^{(12)} = \) the average 12 months performance;
- \( R_i^{(36)} = \) the average 36 months performance.

#### 3.2 Formulation of objective functions by fuzzy interval coefficients

It is impossible to forecast future returns of securities in any budding securities market. The arithmetic mean of historical returns is in general considered as the expected return of the security and so it is obtained as a crisp value. However, for this technique, two main problems need to be solved. If historical data for a long period of time are considered, the influence of the earlier historical data is the same as that of the recent data. However, recent data of a security is more important than the earlier historical data. Secondly, if the historical data of a security are not adequate, due to lack of information (data) the estimation of the statistical parameters are not accurate.

For these reasons, the expected return of a security can be considered as a fuzzy number in place of the arithmetic mean of historical data. Similarly, in a fuzzy environment, the risk and skewness can not be predicted exactly. So the variance and skewness can also...
be considered as fuzzy numbers. For this reason, in this literature the expected return, variance and skewness are considered as fuzzy numbers.

Let the fuzzy expected return, covariance and central co-moments of the $i$th asset are respectively denoted by the following fuzzy numbers: $\tilde{R}_i$, $\tilde{\sigma}_{ij}$, $\tilde{\psi}_{ijk}$.

Then the fuzzy expected return, variance and skewness are respectively defined by

$$
\tilde{R}(x) = \sum_{i=1}^{n} \tilde{R}_ix_i, \quad \tilde{\sigma}^2(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{\sigma}_{ij}x_ix_j, \quad \tilde{S}(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \tilde{\psi}_{ijk}x_ix_jx_k.
$$

So the objectives of the portfolio selection problem are

$$
(3.1) \begin{cases}
\text{Minimize } \tilde{\sigma}^2(x) \\
\text{Maximize } \tilde{R}(x) \\
\text{Maximize } \tilde{S}(x).
\end{cases}
$$

Now we have,

$$
R^L(x, \alpha) = \sum_{i=1}^{n} R^L_i(\alpha)x_i, \quad R^U(x, \alpha) = \sum_{i=1}^{n} R^U_i(\alpha)x_i,
$$

$$
\tilde{\sigma}^2_L(x, \alpha) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma^L_{ij}(\alpha)x_ix_j, \quad \tilde{\sigma}^2_U(x, \alpha) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma^U_{ij}(\alpha)x_ix_j,
$$

$$
S^L(x, \alpha) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \psi^L_{ijk}(\alpha)x_ix_jx_k, \quad S^U(x, \alpha) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \psi^U_{ijk}(\alpha)x_ix_jx_k.
$$

Then by applying the embedding function to problem (3.1), we are with the following optimization problem

$$
(3.2) \begin{cases}
\max \{ R^L(x, \alpha), R^U(x, \alpha), S^L(x, \alpha), S^U(x, \alpha) \} \\
\min \{ \tilde{\sigma}^2_L(x, \alpha), \tilde{\sigma}^2_U(x, \alpha) \}
\end{cases}
$$

**Proposition 3.2.1:** Let $\tilde{R}_i = (R_{i1}, R_{i2}, R_{i3}, R_{i4})$, $\tilde{\sigma}_{ij} = (\sigma_{ij1}, \sigma_{ij2}, \sigma_{ij3}, \sigma_{ij4})$, $\tilde{\psi}_{ijk} = (\psi_{ijk1}, \psi_{ijk2}, \psi_{ijk3}, \psi_{ijk4})$ be all trapezoidal fuzzy numbers. Then,

$$
R^L(x, \alpha) = \sum_{i=1}^{n} R^L_i(\alpha)x_i = \sum_{i=1}^{n} [ R_{i1} + (R_{i2} - R_{i1})\alpha ]x_i,
$$

$$
R^U(x, \alpha) = \sum_{i=1}^{n} R^U_i(\alpha)x_i = \sum_{i=1}^{n} [ R_{i4} + (R_{i4} - R_{i3})\alpha ]x_i,
$$

$$
\tilde{\sigma}^2_L(x, \alpha) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma^L_{ij}(\alpha)x_ix_j = \sum_{i=1}^{n} \sum_{j=1}^{n} [ \sigma_{ij1} + (\sigma_{ij2} - \sigma_{ij1})\alpha ]x_ix_j,
$$

22
\[ \sigma^U_{ij}(x,\alpha) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij}^U(\alpha) x_i x_j = \sum_{i=1}^{n} \sum_{j=1}^{n} \{\sigma_{ij4} - (\sigma_{ij4} - \sigma_{ij3}) \alpha \} x_i x_j, \]

\[ S^L_{ijk}(x,\alpha) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \psi_{ijk}^L(\alpha) x_i x_j x_k = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \{\psi_{ijk1} + (\psi_{ijk2} - \psi_{ijk1}) \alpha \} x_i x_j x_k, \]

\[ S^U_{ijk}(x,\alpha) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \psi_{ijk}^U(\alpha) x_i x_j x_k = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \{\psi_{ijk4} - (\psi_{ijk4} - \psi_{ijk3}) \alpha \} x_i x_j x_k. \]

3.3 The constraints

For the portfolio \( x = (x_1, x_2, \ldots, x_n) \), the expected short term return and long term returns are respectively expressed as

\[ R_{st}(x) = \sum_{i=1}^{n} R_{i(12)} x_i, \]

\[ R_{lt}(x) = \sum_{i=1}^{n} R_{i(36)} x_i. \]

Since investors plan their asset allocation on short-term, long-term or both bases, they should prefer a portfolio having a minimum short-term or long-term or both types of return. For that reason we consider the following two constraints

\[ R_{st}(x) \geq \zeta, \]

\[ R_{lt}(x) \geq \tau, \]

where \( \zeta \) and \( \tau \) will be allocated by the investor.

Dividends are payments made by a company to its shareholders. It is the portion of corporate profits paid out to the investors. For the portfolio \( x = (x_1, x_2, \ldots, x_n) \), the annual dividend is expressed as

\[ D(x) = \sum_{i=1}^{n} d_i x_i. \]

Clearly, investors would like to have a portfolio which yields them a high dividend. Keeping in mind this fact let us propose the following constraint:

\[ D(x) \geq d, \]

where \( d \) will be allocated by the investor.

Liquidity is the degree of probability of converting an investment into cash without any significant loss in value. For an asset liquidity may be measured with respect to the turnover rate. Turnover rate is the proportion between the average stock traded at the market and the tradable stock of that asset. Investors usually prefer greater liquidity.
Turn over rates of assets cannot be predicted precisely. So it is assumed as trapezoidal fuzzy numbers \( \tilde{A}_i = (a_i, b_i, c_i, d_i) \) here. Then the turn over rate of the portfolio \( x = (x_1, x_2, ..., x_n) \) is

\[
\sum_{i=1}^{n} \tilde{A}_i x_i.
\]

The crisp possibilistic mean value of the turn over rate of the portfolio \( x = (x_1, x_2, ..., x_n) \) is given by

\[
L(x) = E\left( \sum_{i=1}^{n} \tilde{A}_i x_i \right) = \int_0^1 (a_i + (b_i - a_i)α) + (d_i - (d_i - c_i)α) dx_i dα = \frac{1}{6} \sum_{i=1}^{n} (a_i + 2(b_i + c_i) + d_i) x_i.
\]

Since investors prefer a portfolio having high liquidity, let us consider the following constraint:

\[
L(x) \geq l,
\]

where \( l \) will be allocated by the investor.

The well known capital budget (sum of proportions invested) constraint on the assets is presented by:

\[
\sum_{i=1}^{n} x_i = 1.
\]

No short selling is considered in the portfolio here. So we have

\[
x_i \geq 0 \quad \forall \ i = 1, 2, ..., n.
\]

### 3.4 The final model

Keeping in mind the objective function and constraints obtained in the previous two subsections, the fuzzy investment problem is constructed as follows:

\[
\begin{align*}
\max \{ & R^L(x, α), R^U(x, α), S^L(x, α), S^U(x, α) \} \\
\min \{ & \sigma^2L(x, α), \sigma^2U(x, α) \} \\
\text{subject to} \ & R_{st}(x) \geq \zeta, \ R_{th}(x) \geq \tau, L(x) \geq l, D(x) \geq d, \ \sum_{i=1}^{n} x_i = 1, x_i \geq 0, \ i = 1, 2, ..., n, α \in [0, 1]. \end{align*}
\]

### 4. Global criteria optimization technique to solve the portfolio selection model

Let the Multi-objective constrained decision making (MOCDM) problem is defined as

\[
\begin{align*}
\max/ \min & \{ f_1(x), f_2(x), ..., f_K(x) \} \\
\text{subject to} \ & x \in X = \{ x : g_s(x) \mid \geq, =, \leq 0, s \in 1, 2, ..., m \}.
\end{align*}
\]
where \( f_a(x) \) are objective functions for maximization, \( a \in A \) and \( f_b(x) \) are objective functions for minimization, \( b \in B, A, B \) being two exhaustive subsets of the index set \( \{1, 2, \ldots, K\} \).

The first step in solving such a problem is to define a reference point. With a given reference point, MCDM problems can be solved by locating the alternatives or decisions closest to the reference point (or the ideal point). Thus the problem becomes how to determine distance to the reference point. The global criteria method measures the distance by using Minkowski’s \( L_p \) metric. The \( L_p \) metric defines distance between two points, \( f \) and \( f^* \) (the reference point) in \( K \)-dimensional space as:

\[
d_p = \left( \sum_k (f_k^* - f_k)^p \right)^{1/p},
\]

where \( p \geq 1 \) and \( k = 1, 2, \ldots, K \).

Unfortunately, because of incommensurability among objects, it is impossible to directly use the above distance family. We need to normalize the distance to remove effects of the incommensurability. Yu and Zeleny (1975) normalized the distance family of Equation (4.2) by using reference points. The distance family becomes:

\[
d_p = \left( \sum_k \left( \frac{f_k^* - f_k}{f_k^*} \right)^p \right)^{1/p},
\]

where \( p \geq 1 \) and \( k = 1, 2, \ldots, K \).

Clearly, when \( p \) increases, the distance \( d_p \) decreases.

To obtain a compromise solution, the global criterion method uses the distance family of Equation (4.3) with the ideal solution being the reference point. The problem is then to solve the following auxiliary problem:

\[
\min_{x \in X} \left( \sum_k \left( \frac{f_k^* - f_k}{f_k^*} \right)^p \right)^{1/p} \quad \text{or} \quad \min_{x \in X} \left( \sum_k \left( \frac{f_k(x^*) - f_k(x)}{f_k(x^*)} \right)^p \right)^{1/p},
\]

where \( x^* \) is the positive ideal solution and \( p = 1, 2, \ldots, \infty \). The value chosen for \( p \) reflects the way of achieving a compromise in minimizing the weighted sum in deviations of criteria from their respective reference points (or ideal solutions). Boychuk and Ovchinnikov (1973) suggested to use \( p = 1 \).

Hwang and Yoon (1981) revise Equation (4.3). Here both PIS(\( f^* \)) and NIS(\( f^- \)) are used to normalize the distance family and obtain

\[
d_p = \left( \sum_k \left( \frac{f_k^* - f_k(x)}{f_k^* - f_k} \right)^p \right)^{1/p},
\]

where \( p \geq 1 \) and \( k = 1, 2, \ldots, K \).

We will use the distance measure given in Equation (4.5) throughout the rest of the paper. Now,
\[ f^* = \{ \max_{x \in X} f_a(x) \land a, \min_{x \in X} f_b(x) \land b \} \]
\[ f^- = \{ \min_{x \in X} f_a(x) \land a, \max_{x \in X} f_b(x) \land b \} \]

where \( a \in A, b \in B \).

Thus \( f^* = \{ f_1^*, f_2^*, \ldots, f_K^* \} \) and \( f^- = \{ f_1^-, f_2^-, \ldots, f_K^- \} \) are a set of positive ideal solutions and a set of negative ideal solutions respectively.

So we are with the following distance measure:

\[
d_p = \left( \sum_{a \in A} w_a^p \left( \frac{f_a^* - f_a(x)}{f_a^* - f_a} \right)^p \right) + \left( \sum_{b \in B} w_b^p \left( \frac{f_b - f_b(x)}{f_b - f_b^*} \right)^p \right)^{1/p}, \tag{4.6} \]

where \( w_k \quad \forall k \) are the weights of the respective objectives.

Then we can solve the MOCDM problem (4.1) by solving the following auxiliary problem:

\[
\begin{aligned}
& \min d_p \\
& \text{such that} \\
& x \in X = \{ x : g_s(x) \geq 0, s = 1, 2, \ldots, m \} \\
& p \geq 1.
\end{aligned} \tag{4.7} \]

When \( p = 1 \), if the objective functions and constraints are all linear, problem (4.7) is simply a linear programming problem and otherwise non-linear. When \( p = 2 \), problem (4.7) becomes a convex programming problem and when \( p = \infty \), problem (4.7) becomes a min-max problem.

**Note:** For the portfolio selection problem (3.3), let us consider 

\[
\begin{aligned}
f_1(x) &= R^L(x, \alpha), \\
f_2(x) &= R^U(x, \alpha), \\
f_3(x) &= S^L(x, \alpha), \\
f_4(x) &= S^U(x, \alpha), \\
f_5(x) &= \sigma^x (x, \alpha), \\
f_6(x) &= \sigma^{\alpha} (x, \alpha).
\end{aligned} \]

So \( A = \{ 1, 2, 3, 4 \}, B = \{ 5, 6 \}, K = 6 \).

Thus we have,

\[
X = \{ x = (x_1, x_2, \ldots, x_n) : R_{s}(x) \geq \zeta, R_{\mu}(x) \geq \tau, L(x) \geq 1, D(x) \geq d, \sum_{i=1}^{n} x_i = 1, x_i \geq 0 \quad \forall i = 1, 2, \ldots, n \}, \tag{4.8}
\]

\[
\begin{aligned}
f^*_1 &= \max_{x \in X} f_1(x), f^*_2 = \max_{x \in X} f_2(x), f^*_3 = \max_{x \in X} f_3(x), \\
f^*_4 &= \max_{x \in X} f_4(x), f^*_5 = \min_{x \in X} f_5(x), f^*_6 = \min_{x \in X} f_6(x), \\
f^-_1 &= \min_{x \in X} f_1(x), f^-_2 = \min_{x \in X} f_2(x), f^-_3 = \min_{x \in X} f_3(x), \\
f^-_4 &= \min_{x \in X} f_4(x), f^-_5 = \max_{x \in X} f_5(x), f^-_6 = \max_{x \in X} f_6(x).
\end{aligned} \tag{4.9}
\]
Applying the above results, the fuzzy MVS portfolio selection problem (3.3) can be converted into the auxiliary problem (4.7).

5. Case study: Application to the stocks of BSE

Bombay Stock Exchange (BSE) is the oldest stock exchange in Asia with a rich heritage of over 133 years of existence. What is now popularly known as BSE was established as "The Native Share & Stock Brokers’ Association" in 1875. It is the first stock exchange in India which obtained permanent recognition (in 1956) from the Government of India under the Securities Contracts (Regulation) Act (SCRA) 1956. With demutualization, the stock exchange has two of world’s prominent exchanges, Deutsche Borse and Singapore Exchange, as its strategic partners. Today, BSE is the world’s number 1 exchange in terms of the number of listed companies and the world’s 5th in handling of transactions through its electronic trading system. The companies listed on BSE command a total market capitalization of USD Trillion 1.06 as of July, 2009.

The BSE Index, SENSEX, is India’s first and most popular stock market benchmark index. Sensex is tracked worldwide. It constitutes 30 stocks representing 12 major sectors. It is constructed on a ‘free-float’ methodology, and is sensitive to market movements and market realities. Apart from the SENSEX, BSE offers 23 indices, including 13 sectoral indices.

In this section we apply our portfolio selection model on the data set extracted from BSE. We have taken monthly share price data for sixty months (March 2003- February 2008) of just five companies which are included in BSE index. Though any number of stocks can be considered, we have taken only 5 stocks to reduce the complexity. Their returns, covariances and central co-moments in the form of fuzzy trapezoidal numbers are used as inputs of the said portfolio optimization problem. If \((a, b, c, d)\) is the fuzzy return of one stock, then \(a\) and \(d\) represent the minimum and the maximum returns of the stock during the 60 months and \([b, c]\) represent the interval containing the most frequent returns. Similar arguments are for variances and central-co moments.

The Table 1 shows the companies name along with their return in the form of trapezoidal fuzzy numbers. The covariances \(\tilde{\sigma}_{ij}\) of the return rates of these risky assets are given in Table 2. The central co-moments are computed and shown in Table 3. The short-term returns, long-term returns, annual dividends and fuzzy turnover rates of the five stocks are given in Table 4.
### Table 1. Fuzzy returns of securities

<table>
<thead>
<tr>
<th>Company</th>
<th>Return ($R_i$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RE</td>
<td>(-0.0080, 0.0223, 0.0501, 0.0673)</td>
</tr>
<tr>
<td>LT</td>
<td>(-0.0031, 0.0287, 0.0611, 0.0866)</td>
</tr>
<tr>
<td>BH</td>
<td>(-0.0020, 0.0282, 0.0581, 0.0832)</td>
</tr>
<tr>
<td>TS</td>
<td>(0.0086, 0.0296, 0.0410, 0.0525)</td>
</tr>
<tr>
<td>SB</td>
<td>(-0.0100, 0.0217, 0.0576, 0.0789)</td>
</tr>
</tbody>
</table>

### Table 2. Fuzzy covariances

<table>
<thead>
<tr>
<th>$\sigma_{11}$</th>
<th>(0.0233, 0.0235, 0.0238, 0.0241)</th>
<th>$\sigma_{22}$</th>
<th>(0.0178, 0.0180, 0.0183, 0.0186)</th>
<th>$\hat{\sigma}<em>{34} = \hat{\sigma}</em>{43}$</th>
<th>(0.0070, 0.0073, 0.0076, 0.0079)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{12} = \sigma_{21}$</td>
<td>(0.0108, 0.0110, 0.0113, 0.0115)</td>
<td>$\sigma_{33} = \sigma_{32}$</td>
<td>(0.0074, 0.0075, 0.0078, 0.0080)</td>
<td>$\hat{\sigma}<em>{35} = \hat{\sigma}</em>{53}$</td>
<td>(0.0045, 0.0048, 0.0050, 0.0053)</td>
</tr>
<tr>
<td>$\sigma_{13} = \sigma_{31}$</td>
<td>(0.0080, 0.0082, 0.0085, 0.0088)</td>
<td>$\sigma_{44} = \sigma_{42}$</td>
<td>(0.0070, 0.0071, 0.0074, 0.0076)</td>
<td>$\hat{\sigma}<em>{45} = \hat{\sigma}</em>{54}$</td>
<td>(0.0180, 0.0183, 0.0186, 0.0189)</td>
</tr>
<tr>
<td>$\sigma_{14} = \sigma_{41}$</td>
<td>(0.0068, 0.0071, 0.0074, 0.0076)</td>
<td>$\sigma_{55} = \sigma_{52}$</td>
<td>(0.0500, 0.0502, 0.0504, 0.05057)</td>
<td>$\hat{\sigma}_{55}$</td>
<td>(0.0122, 0.0126, 0.0129, 0.0131)</td>
</tr>
<tr>
<td>$\sigma_{15} = \sigma_{51}$</td>
<td>(0.0060, 0.0062, 0.0065, 0.0067)</td>
<td>$\sigma_{33}$</td>
<td>(0.0170, 0.0172, 0.0175, 0.0179)</td>
<td>$\sigma_{33}$</td>
<td>(0.0170, 0.0172, 0.0175, 0.0179)</td>
</tr>
</tbody>
</table>

### Table 3. Central product co-moments

<table>
<thead>
<tr>
<th>$\hat{R}_2$</th>
<th>$\hat{R}_4$</th>
<th>$\hat{R}_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{R}_1$</td>
<td>(0.02057, 0.0208, 0.0209, 0.02060)</td>
<td>(0.0157, 0.0158, 0.0159, 0.0160)</td>
</tr>
<tr>
<td>$\hat{R}_2$</td>
<td>(0.00654, 0.00655, 0.00656, 0.00657)</td>
<td>(0.0366, 0.0367, 0.0368, 0.0369)</td>
</tr>
<tr>
<td>$\hat{R}_3$</td>
<td>(0.00654, 0.00655, 0.00656, 0.00657)</td>
<td>(0.0068, 0.0069, 0.0070, 0.0071)</td>
</tr>
<tr>
<td>$\hat{R}_4$</td>
<td>(0.0197, 0.0198, 0.0199, 0.0200)</td>
<td>(0.0366, 0.0367, 0.0368, 0.0369)</td>
</tr>
<tr>
<td>$\hat{R}_5$</td>
<td>(0.00654, 0.00655, 0.00656, 0.00657)</td>
<td>(0.0068, 0.0069, 0.0070, 0.0071)</td>
</tr>
</tbody>
</table>

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Table 4. Short and long term returns, dividends, turnover rates

<table>
<thead>
<tr>
<th>Company</th>
<th>Short term returns ($R_{t}^{(12)}$)</th>
<th>Long-term return ($R_{t}^{(36)}$)</th>
<th>Dividends ($d_{i}$)</th>
<th>Turnover rates ($a_{i}, b_{i}, c_{i}, d_{i}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RE</td>
<td>0.03238</td>
<td>0.032306</td>
<td>63%</td>
<td>(0.0007, 0.0014, 0.0020, 0.0029)</td>
</tr>
<tr>
<td>LT</td>
<td>0.04841</td>
<td>0.048434</td>
<td>85%</td>
<td>(0.0011, 0.0021, 0.0030, 0.0038)</td>
</tr>
<tr>
<td>BH</td>
<td>0.04304</td>
<td>0.043003</td>
<td>125%</td>
<td>(0.0008, 0.0013, 0.0020, 0.0033)</td>
</tr>
<tr>
<td>TS</td>
<td>0.03068</td>
<td>0.030623</td>
<td>155%</td>
<td>(0.0021, 0.0034, 0.0040, 0.0043)</td>
</tr>
<tr>
<td>SB</td>
<td>0.034701</td>
<td>0.034704</td>
<td>140%</td>
<td>(0.0015, 0.0022, 0.0035, 0.0050)</td>
</tr>
</tbody>
</table>

Example 5.1: Corresponding to the set of data in Tables 1, 2, 3 and 4, let us consider the following portfolio selection problem:

$$\max \{ R^{L}(x, \alpha), R^{U}(x, \alpha), S^{L}(x, \alpha), S^{U}(x, \alpha) \}$$

$$\min \{ \sigma^{L}(x, \alpha), \sigma^{U}(x, \alpha) \}$$

subject to

$$R_{il}(x) \geq 0.0373, R_{il}(x) \geq 0.0373,$$

$$L(x) \geq 0.0025, D(x) \geq 1.10,$$

$$x_{1} + x_{2} + x_{3} + x_{4} + x_{5} = 1,$$

$$x_{1}, x_{2}, x_{3}, x_{4}, x_{5} \geq 0,$$

$$0 \leq \alpha \leq 1.$$  

Solution: The global criteria method for multi-objective decision making (as described in section 4) is used to solve the example. From Equation 4.8, we have

$$X = \{ x = (x_{1}, x_{2}, x_{3}, x_{4}, x_{5}) : R_{il}(x) \geq 0.0373, R_{il}(x) \geq 0.0373, L(x) \geq 0.0025, D(x) \geq 1.10, x_{1} + x_{2} + x_{3} + x_{4} + x_{5} = 1, x_{1}, x_{2}, x_{3}, x_{4}, x_{5} \geq 0 \}.$$  

Then the PIS $f^{+}$ and NIS $f^{-}$ of the six objective functions are computed by (4.9) and are shown in Table 5.

For $p \geq 1$ by using Equation (4.10) we get,

$$d_{p} = \left( w_{x}^{p} \left( \frac{0.028775 - R^{L}(x, \alpha)}{0.028775} \right)^{p} + w_{y}^{p} \left( \frac{0.085325 - R^{U}(x, \alpha)}{0.0278387} \right)^{p} + w_{z}^{p} \left( \frac{0.0001 - S^{L}(x, \alpha)}{0.01046} \right)^{p} \right)^{1/p}$$

$$+ w_{w}^{p} \left( \frac{0.0 - S^{U}(x, \alpha)}{0.0104} \right)^{p} + w_{x}^{p} \left( \frac{\sigma^{L}(x, \alpha) - 0.01111942}{0.00255944} \right)^{p} + w_{y}^{p} \left( \frac{\sigma^{U}(x, \alpha) - 0.01157315}{0.000266895} \right)^{p} \right)^{1/p}.$$  

Then by global criteria method, the solution of the portfolio selection problem 5.1 is the same as the solution of the following auxiliary problem:
\[
\begin{align*}
&\min d_p \\
&\text{such that} \\
&x \in X, p \geq 1.
\end{align*}
\]

For different values of \( p \) and \( w_i (i = 1, 2, ..., 6) \), iterations are done and the solutions obtained are shown in Table 6. The different portfolios for different cases are shown in Figure 1.

**Table 5.** PIS\((f^*)\) and NIS\((f^-)\) of the objective functions

<table>
<thead>
<tr>
<th>( f^* )</th>
<th>( f^- )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R^L(x,a) )</td>
<td>0.028775</td>
</tr>
<tr>
<td>( R^L(x,a) )</td>
<td>0.085325</td>
</tr>
<tr>
<td>( \sigma_{2L}(x,\alpha) )</td>
<td>0.01111942</td>
</tr>
<tr>
<td>( \sigma_{2U}(x,\alpha) )</td>
<td>0.01157315</td>
</tr>
<tr>
<td>( S^L(x,\alpha) )</td>
<td>( -0.0001 )</td>
</tr>
<tr>
<td>( S^U(x,\alpha) )</td>
<td>( 0.0 )</td>
</tr>
</tbody>
</table>

**Table 6.** Solution set

<table>
<thead>
<tr>
<th>Case</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>200</td>
<td>2</td>
</tr>
<tr>
<td>( w_1 )</td>
<td>1.0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1.0</td>
</tr>
<tr>
<td>( w_2 )</td>
<td>1.0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1.0</td>
</tr>
<tr>
<td>( w_3 )</td>
<td>0.8</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.8</td>
</tr>
<tr>
<td>( w_4 )</td>
<td>0.8</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.8</td>
</tr>
<tr>
<td>( w_5 )</td>
<td>0.9</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.9</td>
</tr>
<tr>
<td>( w_6 )</td>
<td>0.9</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.9</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>1</td>
<td>0.754</td>
<td>1</td>
<td>0.851</td>
<td>0.758</td>
</tr>
<tr>
<td>( R^L )</td>
<td>0.01827</td>
<td>0.01085</td>
<td>0.01833</td>
<td>0.01402</td>
<td>0.01086</td>
</tr>
<tr>
<td>( R^U )</td>
<td>0.05966</td>
<td>0.06577</td>
<td>0.05966</td>
<td>0.06335</td>
<td>0.06568</td>
</tr>
<tr>
<td>( \sigma_{2L} )</td>
<td>0.0113</td>
<td>0.0113</td>
<td>0.0113</td>
<td>0.0113</td>
<td>0.0113</td>
</tr>
<tr>
<td>( \sigma_{2U} )</td>
<td>0.0116</td>
<td>0.0117</td>
<td>0.0116</td>
<td>0.0116</td>
<td>0.0117</td>
</tr>
<tr>
<td>( S^L )</td>
<td>( -0.0686 )</td>
<td>( -0.0664 )</td>
<td>( -0.0682 )</td>
<td>( -0.0658 )</td>
<td>( -0.0671 )</td>
</tr>
<tr>
<td>( S^U )</td>
<td>( -0.0669 )</td>
<td>( -0.0639 )</td>
<td>( -0.0665 )</td>
<td>( -0.0633 )</td>
<td>( -0.0647 )</td>
</tr>
<tr>
<td>( d_p )</td>
<td>3.23</td>
<td>1.579</td>
<td>3.622</td>
<td>0</td>
<td>1.412</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>0.54028</td>
<td>0.32928</td>
<td>0.53922</td>
<td>0.53192</td>
<td>0.53490</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>0.34684</td>
<td>0.35886</td>
<td>0.34858</td>
<td>0.36062</td>
<td>0.35569</td>
</tr>
<tr>
<td>( x_4 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>0.11288</td>
<td>0.10816</td>
<td>0.11220</td>
<td>0.10746</td>
<td>0.10941</td>
</tr>
</tbody>
</table>

In case 1 of Table 6, highest weights (\( w_1 = w_2 = 1 \)) are given to the return whereas lowest weights (\( w_3 = w_4 = 0.8 \)) are given to asymmetry. Risk is given with medium weights (\( w_5 = w_6 = 1 \)). In this case, the portfolio consists of the stocks of \( LT, BH \) and
SB. 54.028%, 34.684% and 11.288% of the total assets should be invested in the stocks of LT, BH and SB. With this portfolio, the return can be expected to be in $[0.01827, 0.05966]$, the variance in $[0.0113, 0.0116]$, and the skewness in $[-0.0686, -0.0669]$. Similarly the other four cases can be explained.

![Figure 1. Portfolios](image)

6. Conclusions and future work

In this paper, we set up the fuzzy embedding theorem in portfolio selection problem. Then we define MVS portfolio selection problem with constrains on short and long term returns, liquidity and dividends corresponding to the stocks with uncertain return. The efficiency of the portfolios are evaluated by looking for risk contraction on one hand and expected return and skewness augmentation on the other hand. The embedding theorem has been used to convert the fuzzy MVS model into crisp multi-objective multi-criteria decision making problem which is solved by global criteria optimization technique. An empirical application has been tested by the monthly price data set from Bombay stock exchange to illustrate the computational tractability of the approach.

In the future, we will apply these multi-objective fuzzy portfolio selection problem and solution methods to other asset allocation problems, mutual fund portfolio selection, combinational optimization models and multi-period problems. However, the new proposed models of portfolio selection problems and their efficient solution methods will allow us to solve more complicated problems in real situations under more imprecise and ambiguous conditions with chance and possibility/necessity constraints.
References


