Fuzzy Goal Geometric Programming Problem (FG\textsuperscript{2}P\textsuperscript{2}) using Logarithmic Deviational Variables

Payel Ghosh\textsuperscript{*}  
Adamas Institute of Technology, Department of Mathematics, Barasat, P.O.-Jagannathpur, Barbaria,  
24 Parganas (N), West Bengal - 700126, India.  
E-mail: ghoshpayel86@yahoo.com  
\textsuperscript{*}Corresponding Author

Tapan Kumar Roy  
Bengal Engineering and Science University, Department of Mathematics, Shibpur, P.O.-Botanic Garden,  
Howrah, West Bengal-711103, India.  
E-mail: roy_t_k@yahoo.co.in

Received: November 12, 2012 – Revised: February 05, 2013 – Accepted: March 12, 2013

Abstract

The grown interest of non-linear goal programming in various real world problems is showing the essentiality of developments in this context. An efficient method on non-linear goal programming is presented here. A goal programming problem with additive deviational variables is very common where goal programming problem with logarithmic deviational variables is very rare. In this article there is a non-linear goal programming problem with logarithmic deviational variables solved by geometric programming method in imprecise environment. We take the parameters as generalized fuzzy number or more specifically Generalized Trapezoidal Fuzzy Number. Then we solve the fuzzy goal programming problem by geometric programming method with fuzzy parameters. Here in this article the method ‘Fuzzy Goal Geometric Programming Problem (FG\textsuperscript{2}P\textsuperscript{2}) using logarithmic deviational variables’ is applied on a numerical example and an application on ‘Lightly Loaded Bearing Problem’.

Keywords: Goal programming, geometric programming, fuzzy goal programming, fuzzy number, generalized fuzzy number.

1. Introduction

Goal programming is useful for decision maker (DM) to get a prominent decision from multiple objectives. Also it is very difficult for the DM to find a precise target value of each objective function. In that case fuzzy goal programming is introduced. Narasimhan (1980) first introduced fuzzy goal programming (FGP) by using membership functions. Then there are some contributions on FGP with respect to problem formulation, the relative importance, and the fuzzy priority of fuzzy goals. Further Hannan (1985), Rubin (1984), Tiwari et al. (1986), (1987) discussed FGP with weighted method and lexicographic method.

In this paper we have discussed fuzzy goal programming where coefficients are generalized fuzzy number. After the first stone on fuzzy set theory by Zadeh (Bellman and Zadeh, 1970), Chen (1985) developed the theory and applications of generalized fuzzy number. Further developments on arithmetic behavior of generalized fuzzy number had done by Bansal (2011), Garrido (2011), Shridevi et.al. (2009),
Veeramani et al. (2011), Mahapatra and Roy (2011) etc. A particular case on generalized fuzzy number i.e. generalized trapezoidal fuzzy number and its arithmetic operations and applications have been discussed by Banerjee and Roy (2012). We have used the same sense of generalized trapezoidal fuzzy number in goal programming and solved using geometric programming method with fuzzy parameters. The fuzzy geometric programming technique has been introduced by Cao (2002). Further Cao and Yang (2007), (2010) have developed the same.

The paper is organized as follows: At the beginning some key concepts are clarified in abstract. A special type of non-linear crisp goal programming is there followed by introduction. The next in the line is the model formulation of goal programming problem with logarithmic deviational variables. Then there is fuzzy goal geometric programming problem with logarithmic deviational variables followed by some fuzzy preliminaries. An application on “Lightly Loaded Bearing Problem” follows the sequence. After that some conclusions and acknowledgement along with references draws the end of the paper.

2. Crisp Goal Programming

A multi-objective programming can be written as

Find $X = (x_1, x_2 \ldots x_n)^T$

So as to

Minimize $f_{10}(X) = \sum_{i=1}^{P_{10}} C_{10i} \prod_{k=1}^{n} x_k^{a_{10ki}}$ with target value $C_{10}$

Minimize $f_{20}(X) = \sum_{i=1}^{P_{20}} C_{20i} \prod_{k=1}^{n} x_k^{a_{20ki}}$ with target value $C_{20}$

…………………

Minimize $f_{m0}(X) = \sum_{i=1}^{P_{m0}} C_{m0i} \prod_{k=1}^{n} x_k^{a_{m0ki}}$ with target value $C_{m0}$

Subject to $f_r(X) = \sum_{i=1}^{P_r} C_{ri} \prod_{k=1}^{n} x_k^{a_{rki}} \leq C_r, r=1, 2 \ldots q.$

$x_k > 0, k=1, 2 \ldots n$

$C_{j0i}$ and $C_{ri}$ are positive real numbers $\forall$ j, r, i and $a_{j0i}, a_{rki}$ are real numbers $\forall$ k, j, r, i.

$P_{j0} =$ number of terms present in j0-th objective function and $P_r =$ number of terms present in r-th constraint,

$C_r =$ boundary value of r-th constraint.

2.1 Model Formulation of Goal Programming Problem with Logarithmic deviational Variables

Above model of goal programming contains objective functions as “Minimize $f_{j0}(X)$ with target value $C_{j0}$”.

We can take “Minimize $\log (f_{j0}(X))$ with target value $\log (C_{j0})$”.

Also the constraints are $f_r(X) \leq C_r$, i.e. $\log (f_r(X)) \leq \log (C_r)$.

Since the objective functions are to be minimized and constraints are “$\leq$” type, therefore positive deviations should be minimized in each case according to method of goal formulation.

The goal formulation based on weighted sum method is

Minimize $\sum_{j=1}^{m} d_{j0}^+ + \sum_{r=1}^{q} d_{r}^+$

Subject to $\log (f_{j0}(X)) + d_{j0}^+ - d_{j0}^- = \log (C_{j0}), j=1, 2 \ldots m$

$\log (f_r(X)) + d_{r}^+ - d_{r}^- = \log (C_r), r=1, 2 \ldots q$

$x_k > 0, k=1, 2 \ldots n, d_{j0}^+ \times d_{j0}^- = 0, d_{r}^+ \times d_{r}^- = 0,$

$d_{j0}^+, d_{j0}^-, d_{r}^+, d_{r}^- > 0.$

d_{j0}^+ =$ positive deviation of objective function,
\[ d_j^0 = \text{negative deviation of objective function}, \]
\[ d_j^+ = \text{positive deviation of constraint}, \]
\[ d_r^- = \text{negative deviation of constraint}. \]

For a logarithmic change of deviational variables, let \( d_j^0 = \log(u_j^0), d_j^- = \log(u_j^-), d_r^+ = \log(v_r^+), d_r^- = \log(v_r^-) \). Then the above model becomes

\[
\text{Minimize } (\log (\prod u_j^0 \prod v_r^+) )
\]
\[
\text{Subject to } \log (f_j(X) u_j^0) = \log (C_j), j = 1, \ldots, m
\]
\[
\log (f_r(X) v_r^+) = \log (C_r), r = 1, \ldots, q
\]
\[
x_k > 0, k = 1, 2, \ldots, n, u_j^0, u_j^-, v_r^+, v_r^- > 1.
\]

Without loss of generality, the optimal decision variables remain same in the above model and the model

\[
\text{Minimize } \prod u_j^0 \prod v_r^+
\]
\[
\text{Subject to } f_j(X) u_j^0 = C_j, j = 1, \ldots, m
\]
\[
f_r(X) v_r^+ = C_r, r = 1, \ldots, q
\]
\[
x_k > 0, k = 1, 2, \ldots, n, u_j^0, u_j^-, v_r^+, v_r^- > 1.
\]

The goal programming model where the constraints are in inequality form

\[
\text{Minimize } \prod u_j^0 \prod v_r^+
\]
\[
\text{Subject to } f_j(X) u_j^0 \leq C_j, j = 1, \ldots, m
\]
\[
f_r(X) v_r^+ \leq C_r, r = 1, \ldots, q
\]
\[
x_k > 0, k = 1, 2, \ldots, n, u_j^0, u_j^-, v_r^+, v_r^- > 1.
\]

The above goal programming problem can be reduced into weighted goal programming problem with logarithmic deviational variable as

\[
\text{Minimize } \prod_{j=1}^m (u_j^0)^{w_j0} \prod_{r=1}^q (v_r^+)^{w_r}
\]
\[
\text{Subject to } f_j(X) u_j^0^{-1} \leq c_j, j = 1, 2, \ldots, m
\]
\[
f_r(X) v_r^+^{-1} \leq c_r, r = 1, 2, \ldots, q
\]
\[
x_k \geq 0; k = 1, 2, 3, \ldots, n
\]
\[
u_j^0, v_r^+ > 1; \sum_{j=1}^m W_{j0} + \sum_{r=1}^q W_r = 1, W_{j0} > 0, W_r > 0.
\]

A crisp goal programming model with logarithmic deviational variables is solved in Ghosh and Roy (accepted). Here we are going to present a fuzzy goal programming problem with logarithmic deviational variables where the coefficients of each functions and targets of each objective functions are fuzzy and more specifically they are generalized trapezoidal fuzzy number.
3. Fuzzy Preliminaries

**Fuzzy Set:** A fuzzy set \( \tilde{A} \) in a universal set \( X \) is defined as \( \tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X \} \). Here \( \mu_{\tilde{A}}(x) : x \rightarrow [0, 1] \) is the membership value of \( x \) in a fuzzy set \( \tilde{A} \).

**\( \alpha \)-cut of a fuzzy set:** The \( \alpha \) - level set of a fuzzy set \( \tilde{A} \) of \( X \) is a crisp set \( A_\alpha \) containing all the elements of \( x \) that have membership values in \( A \) greater than or equal to \( \alpha \) i.e. \( A_\alpha = \{ x : \mu_{\tilde{A}}(x) \geq \alpha, x \in X, \alpha \in [0,1] \} \).

**Fuzzy Numbers:** A fuzzy number is a connected set of possible values with their own weights from 0 to 1 which does not give any particular value like regular number. The weight is called the membership function.

**Trapezoidal fuzzy number:** A trapezoidal fuzzy number (TrFN) \( \tilde{A} \) is defined as \( (a_1, a_2, a_3, a_4) \) if the membership function

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
0, & x \leq a_1 \\
\frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\
1, & a_2 \leq x \leq a_3 \\
\frac{a_4-x}{a_4-a_3}, & a_3 \leq x \leq a_4 
\end{cases}
\]

**Generalized Fuzzy Number (GFN):** A fuzzy set \( \tilde{A} = (a_1, a_2, a_3, a_4; w) \) defined on the universal set of real numbers \( \mathbb{R} \) is said to be generalized fuzzy number if its membership functions satisfy the following conditions

i) \( \mu_{\tilde{A}}(x) : \mathbb{R} \rightarrow [0,1] \), is continuous,

ii) \( \mu_{\tilde{A}}(x) = 0, \forall x \in (-\infty, a_1] \cup [a_4, \infty) \)

iii) \( \mu_{\tilde{A}}(x) \) is strictly increasing on \([a_1, a_2]\) and strictly decreasing on \([a_3, a_4]\)

iv) \( \mu_{\tilde{A}}(x) = w, \forall x \in [a_2, a_3] \) where \( 0 < w \leq 1 \).

**Generalized Trapezoidal Fuzzy Number (GTrFN):** A generalized fuzzy number is said to be a generalized trapezoidal fuzzy number if its membership function is given by

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
0, & x \leq a_1 \\
w\frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\
w, & a_2 \leq x \leq a_3 \\
w\frac{a_4-x}{a_4-a_3}, & a_3 \leq x \leq a_4 
\end{cases}
\]

and is denoted by \( (a_1, a_2, a_3, a_4; w) \).

**De-fuzzification value of GTrFN:** Let \( \tilde{A} \) be a GTrFN with membership function

\[
\mu_{\tilde{A}}(x) = \begin{cases} 
0, & x \leq a_1 \\
w\frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\
w, & a_2 \leq x \leq a_3 \\
w\frac{a_4-x}{a_4-a_3}, & a_3 \leq x \leq a_4 
\end{cases}
\]
and $\alpha$-cuts $A_\alpha = (a_1 + (a_2 - a_1) \alpha/w, a_4 + (a_4 - a_3) \alpha /w) \ \forall \ \alpha \in [0, w], 0 < w \leq 1.$

Then left removal area $R_l(\bar{A}, 0) = a_2 - \frac{1}{w} \int_{a_1}^{a_2} w \frac{x-a_1}{a_2-a_1} dx$ or $\frac{1}{w} (\int_0^w (a_1 + (a_2 - a_1) \alpha/w) d\alpha) = \frac{a_1 + a_2}{2}$

![Figure 1. Left Removal Area $R_l(\bar{A}, 0)$](image1)

and right removal area $R_r(\bar{A}, 0) = a_3 + \frac{1}{w} \int_{a_3}^{a_4} w \frac{a_4 - x}{a_4 - a_3} dx$ or $\frac{1}{w} (\int_0^w (a_4 + (a_4 - a_3) \alpha/w) d\alpha) = \frac{a_3 + a_4}{2}$

![Figure 2. Right Removal Area $R_r(\bar{A}, 0)$](image2)

The de-fuzzification value of $\bar{A} = (R_l(\bar{A}, 0) + R_r(\bar{A}, 0))/2 = \frac{a_1 + a_2 + a_3 + a_4}{4}$.

4. FG$^2$P$^2$ with logarithmic deviational variables

There are many real life situations where the parameters and goals are ambiguously known to the DM. Keeping this situation under consideration, it is more appropriate to represent the parameters and goals using fuzzy numerical data. Then the goal programming problem using fuzzy parameters and fuzzy goals are viewed as fuzzy goal programming problem. We take the fuzzy parameters and fuzzy goals as GTrFN and then solve the fuzzy goal programming problem using geometric programming method with fuzzy parameters (Yousef et al., 2009). Here is a numerical example of FG$^2$P$^2$ using GTrFN with logarithmic deviational variables.
Minimize \( u^{w_1}v^{w_2} \) \\
subject to \( 1 \ x_1^{-1}x_2^{-2}u^{-1} \leq 4 \)
\( 2 \ x_1^{-2}x_2^{-3}v^{-1} \leq 50 \)
\( x_1 + x_2 \leq 1; \)
\( x_1, x_2 > 0; u > 1, v > 1, w_1, w_2 > 0. \)

Illustration: Let \( \tilde{1} = [1, 1.5, 2, 2.5; h_1 = 0.6], \tilde{2} = [2, 3, 4, 5; h_2 = 0.6], \tilde{3} = [2, 3, 1, 5; h_3 = 0.7], \tilde{50} = [45, 50, 53, 55; h_4 = 0.7]. \)

Let \( \alpha \)-cut of each GTrFN is \( 1_\alpha = (1+0.5 \alpha/0.6, 2.5-0.5 \alpha/0.6) \)
\( 2_\alpha = (1.5+0.5 \alpha/0.6, 3.5-0.5 \alpha/0.6), 4_\alpha = (2+ \alpha/0.7, 5-0.5 \alpha/0.7), 50_\alpha = (45+5 \alpha/0.7, 55-2 \alpha/0.7) \)

1st Sub-problem: \( z^1 = \text{Minimize} \ u^{w_1}v^{w_2} \) \\
subject to \( (1+0.5 \alpha/0.6)(5 - 0.5 \alpha/0.7)^{-1}x_1^{-1}x_2^{-2}u^{-1} \leq 1 \)
\( (1.5+0.5 \alpha/0.6)(55 - 2 \alpha/0.7)^{-1}x_1^{-2}x_2^{-3}v^{-1} \leq 1 \)
\( x_1 + x_2 \leq 1; \)
\( x_1, x_2 > 0, u > 1, v > 1, w_1, w_2 > 0. \)

Here degree of difficulty = \( 5-(4+1) = 0. \)

Taking \( w = \text{Min} \ \{w_1, w_2, w_3, w_4\} = \text{Min} \ \{0.6, 0.6, 0.7, 0.7\} = 0.6 \) dual of (2) is

Maximize \( d(\delta_{01}, \delta_{11}, \delta_{21}, \delta_{31}, \delta_{32}) = \left[ \left( \frac{1}{\delta_{01}} \right)^{\delta_{01}} \times \left( \frac{5+0.5 \alpha/0.6}{2 \delta_{21}} \right)^{\delta_{21}} \times \left( \frac{55-2 \alpha/0.7}{50 \delta_{31}} \right)^{\delta_{31}} \times \left( \frac{1}{\delta_{32}} \right)^{\delta_{32}} \times \lambda_1(\delta) \lambda_2(\delta) \lambda_3(\delta) \right] \) \\
Such that \( \delta_{01} = 1 \)
\( w_1 \delta_{01} - \delta_{11} = 0 \)
\( w_2 \delta_{01} - \delta_{21} = 0 \)
\( -\delta_{11} - 2 \delta_{21} + \delta_{31} = 0 \)
\( -2\delta_{11} - 3 \delta_{21} + \delta_{32} = 0 \)
\( \lambda_1(\delta) = \delta_{11} - \delta_{12} \)
\( \lambda_2(\delta) = \delta_{21} - \delta_{22} \)
\( \lambda_3(\delta) = \delta_{31} + \delta_{32} \)

Solving equation (2.1) – (2.5), \( \delta_{01} = 1, \delta_{11} = w_1, \delta_{21} = w_2, \delta_{31} = w_1+2 \ w_2, \delta_{32} = 2 \ w_1+3 \ w_2, \lambda_1(w_1, w_2) = \delta_{11} = w_1, \lambda_2(w_1, w_2) = \delta_{21} = w_2, \lambda_3(w_1, w_2) = 3 \ w_1+5 \ w_2. \)

From primal dual relation \( x_1x_2^2u = \frac{1+0.5 \alpha/0.6}{5-0.5 \alpha/0.6}, x_1^2x_2^3v = \frac{1.5+0.5 \alpha/0.6}{55-2 \alpha/0.6}, x_1 = \frac{w_1+2 \ w_2}{3 \ w_1+5 \ w_2}, x_2 = \frac{2w_1+3 \ w_2}{3 \ w_1+5 \ w_2}. \)

Let the weight on each function is \( w_1 = 0.5 \) and \( w_2 = 0.5 \) i.e. we are considering equal priority for each objective functions.

Then the first objective function \( f_{11} = x_1^{-1}x_2^{-2}u^{-1} = \frac{5-0.5 \alpha/0.6}{1+0.5 \alpha/0.6}, \) the second objective function
\( f_{21} = x_1^{-2}x_2^{-3}v^{-1} = \frac{55-2 \alpha/0.6}{1.5+0.5 \alpha/0.6}, \) and values of decision variables are \( x_1 = 0.375, x_2 = 0.625. \)
2nd Sub-problem: $z^u = \text{Minimize } u^w_1 v^w_2$

subject to \((2.5-0.5 \alpha/0.6)(2 + \alpha/0.7)^{-1} x_1^{-1} x_2^{-2} u^{-1} \leq 1 \)
\((3.5-0.5 \alpha/0.6)(45 + 5 \alpha/0.6)^{-1} x_1^{-2} x_2^{-3} v^{-1} \leq 1 \)
\(x_1 + x_2 \leq 1; \)
\(x_1, x_2 > 0, u > 1, v > 1, w_1, w_2 > 0.\)

Here degree of difficulty = \(5-(4+1) = 0.\)

Taking \(w = \text{Min} \{w_1, w_2, w_3, w_4\} = \text{Min} \{0.6, 0.6, 0.7, 0.7\} = 0.6\) dual of (3) is

Maximize \[d = \left(\frac{1}{\delta_{01}}\right)^{\delta_{01}} \times \left(\frac{2.5-0.5 \alpha/0.6}{(2+\alpha/0.6)^{\delta_{11}}}\right)^{\delta_{11}} \times \left(\frac{3.5-0.5 \alpha/0.6}{(45+5 \alpha/0.6)^{\delta_{21}}}\right)^{\delta_{21}} \times \left(\frac{1}{\delta_{31}}\right)^{\delta_{31}} \times \left(\frac{1}{\delta_{32}}\right)^{\delta_{32}} \times \lambda_1(\delta) \lambda_2(\delta) \lambda_3(\delta) \]  

Such that \(\delta_{01} = 1\)

\(w_1 \delta_{01} - \delta_{11} = 0\)

\(w_2 \delta_{01} - \delta_{21} = 0\)

\(-\delta_{11} - 2 \delta_{21} + \delta_{31} = 0\)

\(-2 \delta_{11} - 3 \delta_{21} + \delta_{32} = 0\)

\(\lambda_1(\delta) = \delta_{11} - \delta_{12}\)

\(\lambda_2(\delta) = \delta_{21} - \delta_{22}\)

\(\lambda_3(\delta) = \delta_{31} + \delta_{32}\)

Solving equation (3.1) – (3.5), \(\delta_{01} = 1, \delta_{11} = w_1, \delta_{21} = w_2, \delta_{31} = w_1+2 w_2, \delta_{32} = 2 w_1+3 w_2, \lambda_1(w_1, w_2) = \delta_{11} = w_1, \lambda_2(w_1, w_2) = \delta_{21} = w_2, \lambda_3(w_1, w_2) = 3 w_1+5 w_2.\)

From primal dual relation \(x_1 x_2^2 u = \frac{2.5-0.5 \alpha/0.6}{2+\alpha/0.6}, x_1^2 x_2^3 v = \frac{3.5-0.5 \alpha/0.6}{45+5 \alpha/0.6}, x_1 = \frac{w_1+2 w_2}{3 w_1+5 w_2}, x_2 = \frac{2 w_1+3 w_2}{3 w_1+5 w_2}\)

Then the first objective function \(f_{12} = x_1^{-1} x_2^{-2} u^{-1} = \frac{2+\alpha/0.6}{2.5-0.5 \alpha/0.6}\), the second objective function \(f_{22} = x_1^{-2} x_2^{-3} v^{-1} = \frac{45+5 \alpha/0.6}{3.5-0.5 \alpha/0.6}\) and values of decision variables are \(x_1 = 0.375, x_2 = 0.625\) (Taking equal priority for each objective functions).

From the expressions of each objective functions of 1\textsuperscript{st} and 2\textsuperscript{nd} sub-problems, here are the \(\alpha\)-cut of \((f_1)_{\alpha} = (\{\text{Min}(f_{11})_{\alpha}, (f_{12})_{\alpha}\}, \text{Max}\{f_{11})_{\alpha}, (f_{12})_{\alpha}\})\) and \((f_2)_{\alpha} = (\{\text{Min}(f_{21})_{\alpha}, (f_{22})_{\alpha}\}, \text{Max}\{f_{21})_{\alpha}, (f_{22})_{\alpha}\})\) and the corresponding graphs using the software “Mathematica 8” are given in Fig: 3 and Fig: 4.
From Removal area method right removal area of \( f_1(x_1, x_2) \) is \( R_r(f_1, 0) = \frac{1}{0.6} \int_0^{0.6} \left[ \frac{5 - 0.5 \alpha}{0.6} \right] d\alpha = 3.865581 \) (Approx.) and left removal area \( R_l(f_1, 0) = \frac{1}{0.6} \int_0^{0.6} \left[ \frac{2 + 0.5 \alpha}{0.5} \right] d\alpha = 1.124 \) (Approx.)

De-fuzzification value of 1st objective \( \tilde{f}_1(x_1, x_2) = \frac{R_l(f_1, 0) + R_r(f_1, 0)}{2} = 2.4948 \) (Approx.)

From Removal area method right removal area of \( f_2(x_1, x_2) \) is \( R_r(f_2, 0) = \frac{1}{0.6} \int_0^{0.6} \left[ \frac{55 - 2.5 \alpha}{0.6} \right] d\alpha = 31.09721 \) (Approx.) and left removal area \( R_l(f_2, 0) = \frac{1}{0.6} \int_0^{0.6} \left[ \frac{45 + 5 \alpha}{0.6} \right] d\alpha = 10.62 \) (Approx.)

De-fuzzification value of decision variable \( \tilde{f}_2(x_1, x_2) = \frac{R_l(f_2, 0) + R_r(f_2, 0)}{2} = 20.8586 \) (Approx.)
(The calculations of removal areas and de-fuzzification values of decision variables are done using the software Mathematica 8)

Table 1. List of approximated value of decision variables and objective functions

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\hat{f}_1$</th>
<th>$\hat{f}_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.375</td>
<td>0.625</td>
<td>2.4948</td>
<td>20.8586</td>
</tr>
</tbody>
</table>

Note: Here are the de-fuzzified values of objective functions and decision variables.

5. Fuzzy $G^2P^2$ with logarithmic deviational variable on ‘Lightly Loaded Bearing Problem’

A lightly loaded bearing is to be designed to minimize the linear combination of frictional moment, angle of twist of the shaft and the temperature rise of the oil while carrying a load of 1000 lb. Also the ratio of initial fuzzy angular velocity of the shaft and fuzzy angular velocity of the shaft is to be minimized. Assume that 1 in-lb of frictional moment in bearing is equal to 0.0025 rad of angle of twist. There are following goals:

Priority 1: Linear combination of frictional moment, angle of twist of the shaft and the temperature rise of the oil should be minimized and around 10.

Priority 2: Ratio of initial fuzzy angular velocity of the shaft and fuzzy angular velocity of the shaft to be minimized and around 0.8.

Formulate the above goal programming problem and find the dimension of the bearing that is to be built for this purpose such that it can carry maximum load.

Solution: Let R in. be the radius of the journal and L in. be the half length of the bearing. $T$ is the temperature rise of the oil. Then frictional moment of the bearing ($\hat{M}$) = \[ \frac{8\pi\hat{\mu}\hat{\omega}R^2L}{\sqrt{1-\hat{n}^2}} \hat{c} \] where $\hat{\omega}$ is the angular velocity of the shaft, $\hat{\mu}$ is the viscosity of the oil (lubricant), $\hat{n}$ is the eccentricity ratio, $\hat{c}$ is the radial clearance.

Angle of twist of the shaft ($\hat{\phi}$) = $\frac{\hat{S}_c}{G\hat{R}}$ where $\hat{S}_c$ is the shear stress, $\hat{T}$ is the length between the driving point and rotating mass, $G$ is the shear modulus.

The temperature rise of the oil in the bearing is given by $T$ = \[ \frac{0.045\hat{\mu}\hat{\omega}R^2}{c^2n\sqrt{1-n^2}} \]

For the given data $\frac{\hat{c}}{R}$ = [0.0014, 0.0015, 0.0016, 0.0017; 0.7], $\hat{n}$ = [0.8, 0.85, 0.9, 0.95; 0.6] $\hat{\mu}$ = [1/122222, 1/1000000, 1/999999, 1/999989; 0.8] lb-s-in$^{-2}$, $\hat{\omega}$ = [8, 9, 10, 11;0.6] in.$^{-1}$, $\hat{S}_c$ = [29050, 29080, 30000, 30050;0.7] psi, $G$ = 12 $\times$ 10$^6$ psi

Let us choose the initial fuzzy angular velocity $\hat{\omega}_0$ R$^{-1}$ S$^{-1}$.

From the given data in the chart of "Dimension less performance parameters for full journal bearing" assuming the value is not precise $\omega R^{-1}L^3$ = 11.6 i.e. $\hat{\omega}$ = 11.6 R $/$L$^3$.

Let us take $\hat{\omega}_0$ = [92.34, 94.62, 98.04, 101.46; 0.9], $\hat{\omega}$ = [11.4, 11.5, 11.6, 11.7; 0.6]
As per the assumption that 1 in-lb of frictional moment in bearing is equal to 0.0025 rad of angle of twist the linear combination of frictional moment, angle of twist of the shaft and temperature rise of the oil becomes (Taking minimum height of each GTrFN) \( f_1(R, L) = [0.2297, 0.34274, 0.445, 0.6723; 0.6] R^3 L^{-2} + [7.76, 8.72, 10, 11; 0.6] R^{-1} + [0.302, 0.45, 0.59, 0.91; 0.6] RL^{-3} \) with target value \( \tilde{10} \).

\[
\begin{align*}
\text{Ratio of initial fuzzy angular velocity of the shaft and fuzzy angular velocity of the shaft} & = f_2(R, L) = [7.8923, 8.15689, 8.52521, 8.9; 0.6] L^3 \\
\text{with target value} & = 0.8.
\end{align*}
\]

Let us set the target as \( \tilde{10} = [5, 7, 10, 12; 0.6], \tilde{0.8} = [0.7, 0.75, 0.82, 0.87; 0.6] \).

Hence the model of Lightly Loaded Bearing Problem in Fuzzy \( G_2P_2 \) with logarithmic deviational variable is

Minimize \( w_1 u w_2 \) \hspace{1cm} (4.3)

Subject to \( [0.2297, 0.34274, 0.445, 0.6723; 0.6] R^3 L^{-2} u^{-1} + [7.76, 8.72, 10, 11; 0.6] R^{-1} u^{-1} + [0.302, 0.45, 0.59, 0.91; 0.6] R L^{-3} u^{-1} \leq \tilde{10} \)

\( [7.8923, 8.15689, 8.52521, 8.9; 0.6] R^{-1} L^3 v^{-1} \leq \tilde{0.8} \)

\( R, L > 0; u, v > 1; w_1, w_2 > 0 \).

1st Sub-problem:

\( z^1 = \text{Minimize } w_1 u w_2 \) \hspace{1cm} (4.4)

subject to \( (0.2297+0.11304 \alpha/0.6)(12 - 2 \alpha/0.6)^{-1} R^3 L^{-2} u^{-1} + (7.76+0.96 \alpha/0.6)(12 - 2 \alpha/0.6)^{-1} R L^{-3} u^{-1} \leq 1 \)

\( (7.8923+0.26459 \alpha/0.6)(0.87 - 0.05 \alpha/0.6)^{-1} R^{-1} L^3 v^{-1} \leq 1. \)

\( R, L > 0; u, v > 1, w_1, w_2 > 0. \)

Degree of difficulty= 5-(4+1)=0.

Dual of (4.4) is Maximize \( d = \frac{1}{\delta_{01}} \chi \left( \frac{0.2297+0.11304 \alpha/0.6}{(12 - 2 \alpha/0.6)\delta_{11}} \right) \delta_{11} \chi \left( \frac{7.76+0.96 \alpha/0.6}{(12 - 2 \alpha/0.6)\delta_{12}} \right) \delta_{12} \chi \left( \frac{0.302+0.148 \alpha/0.6}{(12 - 2 \alpha/0.6)\delta_{13}} \right) \delta_{13} \chi \left( \frac{7.8923+0.26459 \alpha/0.6}{(0.87 - 0.05 \alpha/0.6)\delta_{21}} \right) \delta_{21} x \lambda_1(\delta) \lambda_1(\delta) x \lambda_2(\delta) \lambda_2(\delta) \) \hspace{1cm} (4.5)

Such that \( \delta_{01} = 1 \)

\( w_1 \delta_{01} - \delta_{11} - \delta_{12} - \delta_{13} = 0 \) \hspace{1cm} (4.6)

\( w_2 \delta_{01} - \delta_{21} = 0 \) \hspace{1cm} (4.7)

\( 3 \delta_{11} - \delta_{12} + \delta_{13} - \delta_{21} = 0 \) \hspace{1cm} (4.8)

\( -2 \delta_{11} - 3 \delta_{13} + 3 \delta_{21} = 0 \) \hspace{1cm} (4.9)

\( \lambda_1(\delta) = \delta_{11} + \delta_{12} + \delta_{13} \) \hspace{1cm} (4.10)

\( \lambda_2(\delta) = \delta_{21} \) \hspace{1cm} (4.11)
Solving equation (4.5) – (4.9), \( \delta_{01} = 1, \delta_{11} = 3(w_1 - w_2)/8, \delta_{12} = 7(w_1 - w_2)/8, \delta_{13} = w_2 - (w_1 - w_2)/4, \delta_{21} = w_2, \lambda_1(w_1, w_2) = \delta_{11} + \delta_{12} + \delta_{13} = w_1, \lambda_2(w_1, w_2) = \delta_{21} = w_2. \)

From primal dual relations we get the following representation of decision variables R and L and also the objective functions\( f_1(R, L), f_2(R, L)\) in terms of \( \alpha \).

\[
R_{11}(\alpha) = \left(\frac{3}{8}\right)^{\frac{1}{8}} \left(\frac{64}{21}\right)^{\frac{1}{8}} \left(\frac{0.2297 + 0.11304 \alpha/0.6}{8w_1}\right)^{\frac{7}{8}} (7.76 + 0.96 \alpha/0.6)^{\frac{7}{8}} \]

\[
L_{11}(\alpha) = \left(\frac{21}{64}\right)^{\frac{1}{8}} \left(\frac{8w_1}{w_1 - w_2}\right)^{\frac{1}{8}} (0.2297 + 0.11304 \alpha/0.6)^{\frac{7}{8}} (7.76 + 0.96 \alpha/0.6)^{\frac{7}{8}} \]

\[
f_{11}(R, L; \alpha) = 0.44 R^3 L^{-2} + 10 R^{-1} + 0.592 RL^{-3} = 0.44 \left(\frac{3}{8}\right)^{\frac{1}{8}} \left(\frac{64}{21}\right)^{\frac{7}{8}} (0.302 + 0.148 \alpha/0.6)^{\frac{7}{8}} \]

\[
(0.2297 + 0.11304 \alpha/0.6)^{\frac{7}{8}} \left(\frac{8w_1}{w_1 - w_2}\right)^{\frac{1}{8}} (7.76 + 0.96 \alpha/0.6)^{\frac{7}{8}} (w_1 - w_2)^{-1} + 10 \left(\frac{3}{8}\right)^{\frac{1}{8}} \left(\frac{64}{21}\right)^{\frac{7}{8}} (0.302 + 0.148 \alpha/0.6)^{\frac{7}{8}} \]

\[
(0.2297 + 0.11304 \alpha/0.6)^{\frac{7}{8}} \left(\frac{8w_1}{w_1 - w_2}\right)^{\frac{1}{8}} (7.76 + 0.96 \alpha/0.6)^{\frac{7}{8}} (w_1 - w_2)^{-\frac{3}{2}}. \]

\[
2^{nd} \text{ Sub-problem:} \]

\[
z^u = \text{Minimize } w_1 v^w_2 \]

subject to \((0.6723 - 0.2273 \alpha/0.6)(5 + 2 \alpha/0.6)^{-1} R^2 L^{-2} u^{-1} + (11- \alpha/0.6)(5 + 2 \alpha/0.6)^{-1} R^{-1} u^{-1} + (0.91 - 0.32 \alpha/0.6)(5 + 2 \alpha/0.6)^{-1} R L^{-3} u^{-1} \leq 1\)

\((8.9 - 0.37479 \alpha/0.6)(0.7 + 0.05 \alpha/0.6)^{-1} R^{-1} L^3 v^{-1} \leq 1.\)

\(R, L > 0; u, v > 1, w_1, w_2 > 0.\)

Degree of difficulty= 5-(4+1)=0.

Dual of (4.4) is Maximize \(d = \left(\frac{1}{\delta_{01}}\right)^{\delta_{10}} x \left(\frac{0.6723 - 0.2273 \alpha/0.6}{(5+2 \alpha/0.6)\delta_{11}}\right)^{\delta_{11}} x \left(\frac{11 - \alpha/0.6}{(5+2 \alpha/0.6)\delta_{12}}\right)^{\delta_{12}} \)

\[
x \left(\frac{0.91 - 0.32 \alpha/0.6}{(5+2 \alpha/0.6)\delta_{13}}\right)^{\delta_{13}} x \left(\frac{8.9 - 0.37479 \alpha/0.6}{(0.7+0.05 \alpha/0.6)\delta_{21}}\right)^{\delta_{21}} x \lambda_1(\delta) \lambda_1(\delta) x \lambda_2(\delta) \lambda_2(\delta) \]

such that \(\delta_{01} = 1\)

\[w_1 \delta_{01} - \delta_{11} - \delta_{12} - \delta_{13} = 0\]

11
\[ w_2 \delta_{01} - \delta_{21} = 0 \]  \hspace{1cm} (4.19)

\[ 3 \delta_{11} - \delta_{12} + \delta_{13} - \delta_{21} = 0 \]  \hspace{1cm} (4.20)

\[ -2 \delta_{11} - 3 \delta_{13} + 3 \delta_{21} = 0 \]  \hspace{1cm} (4.21)

\[ \lambda_1(\delta) = \delta_{11} + \delta_{12} + \delta_{13} \]  \hspace{1cm} (4.22)

\[ \lambda_2(\delta) = \delta_{21} \]  \hspace{1cm} (4.23)

Solving equation (4.17) – (4.21), \( \delta_{01} = 1, \delta_{11} = 3(w_1 - w_2)/8, \delta_{12} = 7(w_1 - w_2)/8, \delta_{13} = w_2 - (w_1 - w_2)/4, \delta_{21} = w_2, \lambda_1(w_1, w_2) = \delta_{11} + \delta_{12} + \delta_{13} = w_1, \lambda_2(w_1, w_2) = \delta_{21} = w_2. \]

From primal dual relations we get the following representation of decision variables \( R \) and \( L \) and also the objective functions \( f_1(R, L), f_2(R, L) \) in terms of \( \alpha \).

\[ R_{12}(\alpha) = \left( \frac{3}{8} \right)^{\frac{1}{2}} \left( \frac{64}{21} \right)^{\frac{1}{2}} \left( \frac{0.6723 - 0.2273 \alpha/0.6}{0.6} \right)^{\frac{3}{8}} \left( 0.91 - 0.32 \alpha/0.6 \right)^{\frac{7}{8}} (11 - \alpha/0.6)^{\frac{7}{8}} \]  \hspace{1cm} (4.24)

\[ L_{12}(\alpha) = \left( \frac{21}{64} \right)^{\frac{1}{2}} \left( \frac{w_1 - w_2}{0.6723 - 0.2273 \alpha/0.6} \right)^{\frac{1}{2}} \left( \frac{5w_2 - 2w_1}{8w_1} \right)^{\frac{1}{2}} (11 - \alpha/0.6)^{\frac{7}{8}} \]  \hspace{1cm} (4.25)

\[ f_{12}(R, L; \alpha) = 0.44 R^3 L^{-2} + 10 R^{-1} + 0.592 RL^{-3} = 0.44 \left( \frac{3}{8} \right)^{\frac{1}{2}} \left( \frac{64}{21} \right)^{\frac{1}{2}} (0.91 - 0.32 \alpha/0.6)^{\frac{7}{8}} \]

\[ (0.6723 - 0.2273 \alpha/0.6)^{-\frac{5}{8}} \left( \frac{5w_2 - 2w_1}{8w_1} \right)^{\frac{1}{2}} (11 - \alpha/0.6)^{\frac{7}{8}} (w_1 - w_2)^{-1} + 10 \left( \frac{3}{8} \right)^{-\frac{1}{2}} \left( \frac{64}{21} \right)^{-\frac{1}{2}} \]

\[ \left( \frac{0.6723 - 0.2273 \alpha/0.6}{0.6} \right)^{\frac{3}{8}} \left( 0.91 - 0.32 \alpha/0.6 \right)^{\frac{7}{8}} (0.6723 - 0.2273 \alpha/0.6)^{-\frac{5}{8}} \left( \frac{5w_2 - 2w_1}{8w_1} \right)^{\frac{1}{2}} (11 - \alpha/0.6)^{\frac{7}{8}} (w_1 - w_2)^{-\frac{3}{2}}. \]  \hspace{1cm} (4.26)

\[ f_{22}(R, L; \alpha) = 8.62 \left( \frac{3}{8} \right)^{\frac{1}{2}} \left( \frac{64}{21} \right)^{-\frac{7}{8}} (0.91 - 0.32 \alpha/0.6)^{\frac{7}{8}} (0.6723 - 0.2273 \alpha/0.6)^{\frac{1}{8}} (11 - \alpha/0.6)^{-\frac{7}{8}} \]

\[ (w_1 - w_2)^{\frac{3}{2}} \left( \frac{5w_2 - 2w_1}{8w_1} \right)^{-\frac{5}{4}}. \]  \hspace{1cm} (4.27)

From the expressions of each objective functions and decision variables of 1\textsuperscript{st} and 2\textsuperscript{nd} sub-problems, here are the \( \alpha \)-cut of \( (f_1)_{a} = (\text{Min}\{(f_{11})_{a}, (f_{12})_{a}\}), \text{Max}\{(f_{11})_{a}, (f_{12})_{a}\}), (f_2)_{a} = (\text{Min}\{(f_{21})_{a}, (f_{22})_{a}\}), \text{Max}\{(f_{21})_{a}, (f_{22})_{a}\}), R_{a}=(\text{Min}\{(R_{11})_{a}, (R_{12})_{a}\}), \text{Max}\{(R_{11})_{a}, (R_{12})_{a}\}), L_{a}=(\text{Min}\{(L_{11})_{a}, (L_{12})_{a}\}), \text{Max}\{(L_{11})_{a}, (L_{12})_{a}\}) \) taking weight \( w_1=0.6, w_2=0.4 \) and the corresponding graphs using the software “Mathematica 8” are given in Fig: 5, Fig: 6, Fig: 7, Fig: 8.
Figure 5. Rough sketch of $\alpha$-cut of $f_1(R, L)$

From Removal area method right removal area is $R_r(f_1, 0) = \frac{1}{0.6} \int_0^{0.6} (0.44 \frac{3}{8})^2 (\frac{64}{21})^2 (0.302 + 0.148 \alpha_{0.6})^{-\frac{1}{4}} \left(0.2297 + 0.11304 \alpha_{0.6} \right)^{-\frac{3}{8}} \left(\frac{5w_2 - 2w_1}{8w_1}\right)^{\frac{7}{8}} (w_1 - w_2)^{-1} + 10(\frac{3}{8})^{-\frac{1}{2}} (\frac{64}{21})^{-\frac{1}{8}} \left(0.2297 + 0.11304 \alpha_{0.6}\right)^{\frac{3}{8}} (\frac{5w_2 - 2w_1}{8w_1})^\frac{5}{8} (7.76 + \frac{0.96\alpha_{0.6}}{0.06}) \frac{7}{8} (w_1 - w_2)^{-\frac{3}{7}} d\alpha$

= 18.75413 (Approx.)

and left removal area $R_l(f_1, 0) =$

\[
\frac{1}{0.6} \int_0^{0.6} (0.44 \frac{3}{8})^2 (\frac{64}{21})^2 (0.91 - 0.32 \alpha_{0.6})^{-\frac{1}{4}} \left(0.6723 - 0.2273 \alpha_{0.6} \right)^{-\frac{5}{8}} \left(\frac{5w_2 - 2w_1}{8w_1}\right)^{\frac{1}{8}} (11 - \alpha_{0.6})^\frac{7}{8} (w_1 - w_2)^{-1} + 10(\frac{3}{8})^{-\frac{1}{2}} (\frac{64}{21})^{-\frac{1}{8}} \left(0.6723 - 0.2273 \alpha_{0.6}\right)^{\frac{3}{8}} (\frac{5w_2 - 2w_1}{8w_1})^\frac{5}{8} (11 - \alpha_{0.6}) \frac{3}{8} \frac{7}{8} (w_1 - w_2)^{-\frac{3}{7}} d\alpha
\]

= 16.6932 (Approx.).

De-fuzzification value of decision variable $\tilde{f}_1 = (R_l(f_1, 0) + R_r(f_1, 0))/2 = 17.72$ (Approx.)
From Removal area method left removal area is
\[
R_l(\hat{f}_2, 0) = \frac{1}{0.6} \int_0^{0.6} \left\{ 8.62 \left( \frac{3}{8} \right)^7 \left( \frac{64}{21} \right)^7 \left( 0.302 + 0.148 \frac{\alpha}{0.6} \right)^{5/4} \left( 7.76 + 0.6 \right)^{5/4} \left( w_1 - w_2 \right)^{3/2} \left( \frac{5w_2 - 2w_1}{8w_1} \right)^{5/4} \} \, d\alpha = 0.335787 \text{ (Approx.)}
\]
and right removal area \( R_r(\hat{f}_2, 0) = \frac{1}{0.6} \int_0^{0.6} \left\{ 8.62 \left( \frac{3}{8} \right)^7 \left( \frac{64}{21} \right)^7 \left( 0.91 - 0.32 \frac{\alpha}{0.6} \right)^{5/4} \left( 0.6723 - 0.2273 \frac{\alpha}{0.6} \right) \left( w_1 - w_2 \right)^{3/2} \left( \frac{5w_2 - 2w_1}{8w_1} \right)^{5/4} \right\} \, d\alpha = 0.700561 \text{ (Approx.)}

Defuzzification value of decision variable \( \hat{f}_2(R, L) = \frac{R_l(\hat{f}_2, 0) + R_r(\hat{f}_2, 0)}{2} = 0.518174 \text{ (Approx.)} \)
From Removal area method right removal area of R is

\[ R_r(\hat{R}, 0) = \frac{1}{0.6} \int_0^{0.6} \left\{ \left( \frac{3}{8} \right)^{\frac{1}{2}} \left( \frac{64}{21} \right)^{\frac{1}{2}} \frac{(0.2297+0.11304 \alpha/0.6)^{\frac{1}{2}}}{(0.302+0.148 \alpha/0.6)^{\frac{1}{2}} (7.76+0.96\alpha/0.6)^{\frac{1}{2}}} \right\} \, d\alpha = 2.0412 \text{ (Approx.)} \]

and left removal area \( R_l(\hat{R}, 0) \) is

\[ R_l(\hat{R}, 0) = \frac{1}{0.6} \int_0^{0.6} \left\{ \left( \frac{3}{8} \right)^{\frac{1}{2}} \left( \frac{64}{21} \right)^{\frac{1}{2}} \frac{(0.6723-0.2273 \alpha/0.6)^{\frac{1}{2}}}{(0.91-0.32 \alpha/0.6)^{\frac{1}{2}} (11-\alpha/0.6)^{\frac{1}{2}}} \right\} \, d\alpha = 1.94559 \text{ (Approx.)} \]

De-fuzzification value of decision variable \( \hat{R} = (R_l(\hat{R}, 0) + R_r(\hat{R}, 0))/2 = 1.9934 \text{ (Approx.)} \)

From Removal area method left removal area of L is

\[ R_l(\hat{L}, 0) = \frac{1}{0.6} \int_0^{0.6} \left\{ \left( \frac{21}{64} \right)^{\frac{1}{2}} \frac{(w_1-w_2)^{\frac{1}{2}} (0.302+0.148 \alpha/0.6)^{\frac{1}{2}}}{(0.2297+0.11304 \alpha/0.6)^{\frac{1}{2}} (5w_2-2w_1)^{\frac{1}{2}}} \right\} \, d\alpha = 0.05289 \text{ (Approx.)} \]

and right removal area \( R_r(\hat{L}, 0) \) is

\[ R_r(\hat{L}, 0) = \frac{1}{0.6} \int_0^{0.6} \left\{ \left( \frac{21}{64} \right)^{\frac{1}{2}} \frac{(w_1-w_2)^{\frac{1}{2}} (0.91-0.32 \alpha/0.6)^{\frac{1}{2}}}{(0.6723-0.2273 \alpha/0.6)^{\frac{1}{2}} (5w_2-2w_1)^{\frac{1}{2}}} \right\} \, d\alpha = 0.0594629 \text{ (Approx.)} \]

De-fuzzification value of decision variable \( \hat{L} = (R_l(\hat{L}, 0) + R_r(\hat{L}, 0))/2 = 0.0561785 \text{ (Approx.)} \)

(The calculations of removal areas and de-fuzzification values of decision variables are done using the software Mathematica 8)

**Table 2.** List of approximated value of decision variables and objective functions

<table>
<thead>
<tr>
<th>( \hat{R} ) (in.)</th>
<th>( \hat{L} ) (in.)</th>
<th>( \hat{f}_1 )</th>
<th>( \hat{f}_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.9934</td>
<td>0.0561785</td>
<td>17.72</td>
<td>0.518174</td>
</tr>
</tbody>
</table>

Note: Here are the de-fuzzified values of objective functions and decision variables
6. Conclusion

In this paper we have described an imprecise method of solving multi-objective non-linear programming problem. A multi-objective non-linear programming can be written as goal programming problem and solved by geometric programming technique where coefficients are fuzzy. We take all the coefficients and targets as GTrFN and solved the goal programming problem using geometric programming problems with fuzzy parameters. As the deviations are taken in logarithmic sense therefore the degree of difficulty in geometric programming is less than other method of goal programming.

Acknowledgements

It’s my privilege to thank my respected guide for his enormous support and encourage for the preparation of this research paper. The great support of my family members makes me possible to do it.

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